MATH 245 S21, Exam 3 Solutions

2. Every solution should be unique, because they all begin by choosing your personal favorite three real numbers. For these solutions, we will use $a = 1, b = 0, c = \pi$, hence $S = \{1, 0, \pi\}, T = \{1, 1 + \pi\}$.

The exam instructions said "it would be unwise to present someone else's favorite real numbers as your own." Half the class unwisely used $a = 1, b = 2, c = 3$. Some students innocently came up with these exact numbers, in that exact order, on their own; however, others were communicating during the exam. Picking $1, 2, 3$ is not by itself proof of anything, but it is suspicious. Those who did so now have evidence against them, perhaps in addition to other accumulated evidence, and are that much closer to serious cheating penalties and mandatory reporting. Be careful!

3. Let S, T be as defined in Question 2, and $R = \{x \in \mathbb{R} : 1 < x^2 \leq 10\}$. Prove or disprove that $S \subseteq R \cup T$.

The statement is false; to disprove we need a counterexample. $0 \in S$. $0 \notin R$ (since $0^2 = 0$ is not in the interval $(1, 10]$), and also $0 \notin T = \{1, 1 + \pi\}$. Hence $0 \notin R \cup T$. Note: the statement might be true for *your* S, T.

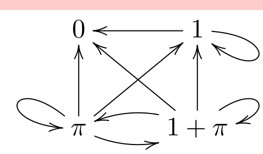
4. Let S, T be as defined in Question 2. (i) Find any nonempty $R_1 \subseteq S\Delta T$; (ii) Find any nonempty $R_2 \subseteq S \times T$; and (iii) Find any partition of $S \times T$.

This question is partly about definitions of symmetric difference, Cartesian product, partition and partly about notation and categories. It is important that R_1, R_2 be sets (denoted with curly brackets), whose elements are real numbers (for R_1) or ordered pairs (for R_2). It is important that the partition is a set of sets, whose elements are ordered pairs. For every choice of S, T, there are many possible answers to all three subquestions.

One possible collection of correct answers for my S, T is $R_1 = \{0, 1 + \pi\}, R_2 = \{(1, 1), (0, 1 + \pi)\}$, and partition $\{\{(1, 1), (1, 1 + \pi), (0, 1)\}, \{(0, 1 + \pi)\}, \{(\pi, 1), (\pi, 1 + \pi)\}\}$.

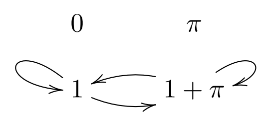
5. Let S, T be as defined in Question 2. Consider relation R on $S \cup T$ given by $R = \{(x, y) : x \geq |y - 1|\}$. Draw this relation as a digraph, and determine whether or not it is antisymmetric.

Your digraph should have a loop at every vertex x with $x \geq 0.5$. For $x, y \geq 1$, we have both (x, y) and (y, x) in R if $x + 1 \geq y \geq x - 1$, i.e. y and x are within 1 of each other. Hence, for my S, T, the digraph is not antisymmetric because π and $1 + \pi$ each connect with the other. The digraph for my S, T appears at right.



6. Let S, T be as defined in Question 2. Find a relation R on $S \cup T$ that is symmetric, not reflexive, and not trichotomous, but where $R|_T$ is reflexive and trichotomous. Give R both as a set and as a digraph.

To make $R|_T$ reflexive and trichotomous, we must include $(1, 1), (1 + \pi, 1 + \pi)$, and at least one of $\{(1, 1 + \pi), (1 + \pi, 1)\}$ in R. However, to make R symmetric, in fact we must include both $(1, 1 + \pi)$ and $(1 + \pi, 1)$. Hence the simplest solution is $R = \{(1, 1), (1 + \pi, 1 + \pi), (1, 1 + \pi), (1 + \pi, 1)\}$ with digraph as shown to the right.



7. This problem no longer uses S, T from Question 2. Prove or disprove: For all sets A, B, C, we must have $B \cap C \subseteq (A \setminus B)\Delta C$.

The statement is true. Let A, B, C be arbitrary sets. Let $x \in B \cap C$ be arbitrary. Hence $x \in B \wedge x \in C$, and by simplification we get each of $x \in B$ and $x \in C$. Now, by addition with $x \in B$, we get $x \notin A \vee x \in B$. By De Morgan's Law (for propositions), we get $\neg(x \in A \wedge x \notin B)$. Hence $\neg x \in A \setminus B$, i.e. $x \notin A \setminus B$. We now use conjunction with $x \in C$, to get $(x \notin A \setminus B \wedge x \in C)$. Next, we use addition to get $(x \in A \setminus B \wedge x \notin C) \vee (x \notin A \setminus B \wedge x \in C)$. Lastly, we conclude $x \in (A \setminus B)\Delta C$.

8. This problem no longer uses S, T from Question 2. Prove or disprove: For all sets A, B, we must have $2^A \Delta 2^{(A\Delta B)} = 2^B$.

The statement is false, so we need a counterexample. Take $A = \{a, b\}, B = \{b, c, d\}$, so $A\Delta B = \{a, c, d\}$.

SOLUTION 1: Take $x = \{a, c\}$. We have $x \subseteq A\Delta B$, and hence $x \in 2^{(A\Delta B)}$. However, $x \not\subseteq A$ so $x \notin 2^A$. By conjunction $x \notin 2^A \wedge x \in 2^{(A\Delta B)}$, and by addition $(x \notin 2^A \wedge x \in 2^{(A\Delta B)}) \vee (x \in 2^A \wedge x \notin 2^{(A\Delta B)})$. Hence $x \in 2^A \Delta 2^{(A\Delta B)}$. But $x \not\subseteq B$ so $x \notin 2^B$. Hence $2^A \Delta 2^{(A\Delta B)} \not\subseteq 2^B$.

SOLUTION 2: Take $x = \{b, c\}$. We have $x \subseteq B$ so $x \in 2^B$. We also have $x \not\subseteq A\Delta B$, so $x \notin 2^{(A\Delta B)}$, and $x \not\subseteq A$, so $x \notin 2^A$. By conjunction, $x \notin 2^A \wedge x \notin 2^{(A\Delta B)}$, and by De Morgan's Law $\neg(x \in 2^A \vee x \in 2^{(A\Delta B)})$. Hence $\neg x \in (2^A \cup 2^{(A\Delta B)})$, i.e. $x \notin (2^A \cup 2^{(A\Delta B)})$. By addition $x \notin (2^A \cup 2^{(A\Delta B)}) \vee x \in (2^A \cap 2^{(A\Delta B)})$, and by De Morgan's Law $\neg(x \in (2^A \cup 2^{(A\Delta B)}) \wedge x \in (2^A \cap 2^{(A\Delta B)}))$, i.e. $\neg x \in (2^A \cup 2^{(A\Delta B)}) \setminus (2^A \cap 2^{(A\Delta B)})$. We now apply Theorem 8.12 to conclude $x \notin (2^A \Delta 2^{(A\Delta B)})$.

SOLUTION 3: It is also possible to simply write out $2^A \Delta 2^{(A\Delta B)}$ as a set of sets (by first calculating 2^A, $2^{(A\Delta B)}$), and 2^B as another set of sets, and find an explicit set that's in one but not the other.